2000-2001 W/SOLUTIONS

Test #1 **AMATYC Student Mathematics League** October-November 2000

1. A line passes through the point (-3, -2). What is the greatest slope it could have if it never enters the second quadrant (the axes do not belong to any quadrant)?

   A. $\frac{3}{2}$ B. $\frac{-3}{2}$ C. $\frac{2}{3}$ D. $\frac{-2}{3}$ E. 0

2. If $i = \sqrt{-1}$, then $4(1 + i)^{-1}$ is

   A. $2 + 2i$ B. $2 - 2i$ C. $4 + 4i$ D. $4 - 4i$ E. undefined

3. The area of the triangle with vertices (1, -2), (9, 2), and (5, 5) is

   A. 18 B. 20 C. 22 D. 24 E. 26

4. A banquet hall has capacity 400 persons (including both diners and servers). If one server is needed for every 12 diners, the maximum number of diners is

   A. 366 B. 367 C. 368 D. 369 E. 370

5. The letters A, M, A, T, Y, C are listed so that letters are in increasing order of the number of distinct line segments or curves used to draw them, and identical letters are not adjacent. The fourth letter of the list is

   A. A B. C C. M D. T E. Y

6. The sum of all solutions of the equation $|2x - 1| = 7$ is

   A. 4 B. 3 C. 2 D. 1 E. 0

7. Two runners running around a 600 m track in opposite directions and starting from the same place run a lap in 100 sec and 150 sec respectively. How many meters from their starting place are the runners when they meet for the eighth time (not counting the start)?

   A. 60 B. 120 C. 180 D. 240 E. 300

8. Electrical resistance in a wire is directly proportional to its length and inversely proportional to the square of its diameter. If a 10 cm long wire with diameter 2 cm has resistance 600 ohms, a 15 cm long wire with diameter 5 cm has resistance

   A. 3750 ohms B. 2500 ohms C. 1000 ohms D. 360 ohms E. 144 ohms

9. If $\cot C = 2$, then $\sin 2C =$

   A. $\frac{3}{\sqrt{10}}$ B. $\frac{1}{\sqrt{5}}$ C. $\frac{2}{\sqrt{5}}$ D. $\frac{4}{5}$ E. $\frac{9}{10}$

10. An ice cream parlor sells cones with at most 3 scoops, each scoop a different flavor chosen from vanilla, chocolate, or strawberry. If a customer randomly selects one of the parlor's possible kinds of cones (order of the flavors doesn't matter), the probability that the cone includes a scoop of chocolate is

    A. $\frac{1}{3}$ B. $\frac{3}{8}$ C. $\frac{3}{7}$ D. $\frac{1}{2}$ E. $\frac{4}{7}$

11. ΔABC is a right triangle with hypotenuse $\overline{AC}$, AB = 8, and BC = 6. Point D is chosen on $\overline{BC}$ so that BD = 5, and ray BA is extended to point E so that BE = 12. If F is the intersection of $\overline{AC}$ and $\overline{DE}$, the distance DF equals

    A. $\frac{13}{4}$ B. $\frac{15}{4}$ C. 4 D. $\frac{25}{4}$ E. $\frac{39}{4}$

12. To the nearest tenth, the sum of the nonreal solutions of the equation $x^3 + 3x^2 - 5 = 0$ is

    A. -4.1 B. -3.0 C. 0.0 D. 1.9 E. 3.9

13. How many solutions in the interval $-2\pi \le t \le 2\pi$ does the equation $\cos 2t = \sin t + \cos t$ have?

A. 4 B. 5 C. 7 D. 9 E. 10

14. Which of the following best describes the set of real values of r for which there exist a real value of s such that $\log r + \log s = \log (r + s)$?

A. all r B. all $r > 0$ C. all r between 0 and 1 D. all $r > 1$ E. no r

15. Rectangle R_1 has vertices $(\pm 5, \pm 4)$, and rectangle R_2 has vertices $(\pm 3, \pm 2)$. The probability to the nearest hundredth that a point chosen at random from the interior of R_1 is no more than 1 unit from some point of R_2 is

A. 0.40 B. 0.45 C. 0.49 D. 0.50 E. 0.59

16. An object moves along the number line from 0 to 10 by moving either 1 or 2 units each time it moves. How many different sequences of moves are possible?

A. 9 B. 10 C. 32 D. 55 E. 89

17. If (x, y) is any point in the solution set of the system $\begin{cases} 2x + 3y \le 42 \\ 3x + 2y \ge 24 \\ 2x - y \ge 2 \\ x - 2y \le 0 \end{cases}$, the smallest possible value of $2x + y$ is

A. 12 B. 14 C. 15 D. 22 E. 30

18. Two vehicles leave an intersection at the same time, one headed northwest at 30 mph, the other headed east at 40 mph. To the nearest 1 mph, how fast are they moving apart?

A. 41 mph B. 50 mph C. 55 mph D. 58 mph E. 70 mph

19. Which of the numbers 2, 3, or 4 is a factor of $5^{2000} - 1$?

A. 2 only B. 3 only C. 2 and 4 only D. 2 and 3 only E. 2, 3, and 4

20. In a collection of 2000 positive integers, the sum of the mean μ, median M, and (unique) mode m is 5. Which of the following is possible?

A. $M = m = 2$ B. $\mu = M = 1$ C. $\mu = M = 2$ D. $M = m = 1$ E. $\mu = m = 1$

Test #2 **AMATYC Student Mathematics League** **February 2001**

1. Let M be the line with equation $3x - 2y = 9$, and let L be the line perpendicular to M and having the same x-intercept as M. Then the y-intercept of L is

   A. -2 B. 2 C. 4 D. 4.5 E. 6

2. Which of the following points is closest to the point (4, -5)?

   A. (-9, 3) B. (-11, -2) C. (-4, 8) D. (-10, 1) E. (1, 10)

3. What value of b will make 2 a solution of the equation $2(x - b) + (x + b) = 3(b - x)$?

   A. 3 B. 1 C. -1 D. -3 E. any value of b

4. If $f(x) = x^2 - 2$, $g(x) = 2x + 1$, and $h(x) = x - 1$, then which of the following equals 4 ?

   A. $f(g(h(2)))$ B. $f(h(g(2)))$ C. $g(f(h(2)))$ D. $g(h(f(2)))$ E. $h(g(f(2)))$

5. For t measured in radians, $\cos\left(\frac{3\pi}{2} - t\right) =$

   A. cos t B. - cos t C. sin t D. - sin t E. none of these

6. The sum of $\log_4 8$ and $\log_4 32$ is closest to

   A. 2.66 B. 3 C. 3.75 D. 4 E. undefined

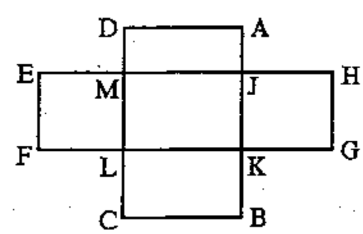
7. The domain of the function $f(x) = \dfrac{x - \dfrac{3}{x - 2}}{\dfrac{x + 1}{x - 1}}$ is all real numbers x except

   A. 2 only B. 1 only C. -1 only D. 1 and 2 only E. -1, 1, and 2 only

8. In the accompanying diagram, ABCD and EFGH are both rectangles, with $AB = 10$, $BC = 6$, $EF = 4$ and $FG = 15$. If $\overline{EF} \parallel \overline{AB}$, then the area enclosed by the polygon AJHGKBCLFEMD is

   A. 72 B. 96 C. 100 D. 120

   E. dependent on the choice of J, K, L, and M



9. For how many values of k does the system $\begin{cases} x + ky = 3 \\ kx + 2y = 2 \\ x + y = 1 \end{cases}$ have at least one solution?

   A. none B. 1 C. 2 D. 3 E. 4

10. Call a date *lucky* if the number of the month is a factor of the number of the day. For example, Valentine's Day, February 14, is lucky (2 is a factor of 14), but Christmas Day, December 25, is not (12 is not a factor of 25). If a day is randomly selected from the year, which of the following is closest to the probability that the day is lucky?

    A. 0.16 B. 0.20 C. 0.24 D. 0.25 E. 0.30

11. The equation $\dfrac{x^4 - 2x^3 + 2x^2 - 2x + 1}{x - 1} = 0$ has how many different solutions, real or imaginary?

    A. 0 B. 1 C. 2 D. 3 E. 4

12. Let S be the set of all points in the plane which are twice as far from the point (3,-1) as from the point (4,2). The points of S lie on what curve?

A. a straight line B. a circle C. a parabola D. a hyperbola E. an ellipse which is not a circle

13. Al and Bo are each either a knight, who always tells the truth, a knave, who always lies, or an alternator, who lies and tells the truth in strict alternation. Al says, "I am an alternator," Bo says, "Al is a knave," and then Al says, "Bo is not an alternator." Which of the following must be true?

A. Al is a knave B. Al is an alternator C. Bo is a knight

D. Bo is a knave E. Bo is an alternator

14. Let AMA and TYC be three-digit numbers (a leading digit cannot be zero) such that each different letter represents a different decimal digit 0 to 9. If AMA − TYC is 22, what is the value of T - C?

A. -1 B. 0 C. 1 D. 2 E. it depends on the choices of A and M

15. In how many ways can the expression a + a + b + b + b + c + c + c be written (including the expression itself) using the Commutative Property of Addition without combining any like terms?

A. 108 B. 560 C. 1120 D. 3360 E. 40320

16. If $\overline{CM}$ is the median to side $\overline{AB}$ in ΔABC, with AB = 9, AC = 11, and BC = 8, then CM =

A. 8.5 B. 9 C. 9.5 D. 10 E. 10.5

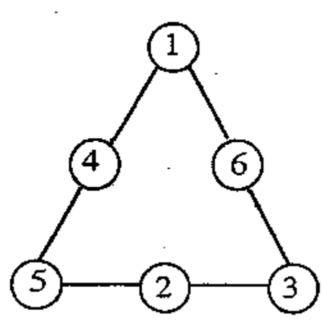
17. Define a sequence a_n by $a_1 = 1$ and for $n \ge 1$, $a_{2n} = 2a_{2n-1}$, $a_{2n+1} = a_{2n} + 1$. Which of the following is closest to $\sqrt[2001]{a_{2001}}$? A. 1 B. $\sqrt{2}$ C. 2 D. $2\sqrt{2}$ E. 3

18. A collection of seven positive integers has median 3 and unique mode 4. If the collection is increased to nine values by adding two 2's to it, the median and unique mode both are 2. What is the mean of the new collection?

A. $\frac{23}{9}$ B. $\frac{8}{3}$ C. $\frac{25}{9}$ D. $\frac{26}{9}$ E. 3

19. How many pairs (r, s) of positive integers satisfy the equation $r^2 - s^2 = 2001$?

A. 0 B. 1 C. 2 D. 3 E. 4

20. 

The numbers 1, 2, 3, 4, 5, and 6 are each placed at a different point on an equilateral triangle, either a vertex or a midpoint of a side, in such a way that the sums of the three numbers on each side are all equal (An example with a common sum of 10 is shown at the left). How many different sums are possible?

A. 1 B. 2 C. 3 D. 4 E. 5

1. If $f(x) = x^2 - 5$ and $g(x) = 2x - 3$, which of the following is true about $r = f(-2)$, $s = g(-2)$, and $t = f(g(-2))$?

   A. $t < r < s$ B. $s < r < t$ C. $s < t < r$ D. $r < s < t$ E. $r < t < s$

2. For $1 \le x \le 2$, which of the following is equal to $\big||x - 3| - |1 - x|\big|$?

   A. $2x - 4$ B. $4 - 2x$ C. $2 - 2x$ D. 2 E. 4

3. A line has slope 2, and the sum of its intercepts is k. If the line is shifted vertically upward by 4 units, the sum of the intercepts of the shifted line is

   A. $k + 2$ B. $k + 4$ C. $k + 8$ D. $k - 2$ E. $k - 4$

4. Each day Carlos makes an open-faced sandwich (that is, with only one slice of bread) with either one kind of meat or one kind of cheese or both. If he has four kinds of bread, five kinds of meat, and three kinds of cheese, how many different sandwiches can he make?

   A. 60 B. 72 C. 80 D. 92 E. 96

5. George invests a certain amount of money in the stock market. After his investment increases by 10%, he takes out \$150 and sets it aside. His remaining investment then decreases by 10%. If George's total (including the \$150 set aside) shows a net increase of 4%, by how much has his total investment increased?

   A. \$6 B. \$10 C. \$12 D. \$16 E. \$20

6. Call a date *lucky* if the number of the month is a factor of the number of the day. For example, Valentine's Day, February 14, is lucky (2 is a factor of 14), but Christmas Day, December 25, is not (12 is not a factor of 25). What is the greatest number of consecutive months having equal numbers of lucky days?

   A. 1 B. 2 C. 3 D. 4 E. 5

7. If AM + AT = YC, where each different letter represents a different digit and each letter pair represents a two-digit number, what is the largest possible value of YC?

   A. 89 B. 91 C. 95 D. 97 E. 98

8. The domain of the real-valued function $f(x) = \sqrt{4 - \sqrt{4 - x}}$ is

   A. $x \ge 4$ B. $4 \le x \le 20$ C. $0 \le x \le 4$ D. $-4 \le x \le 4$ E. $-12 \le x \le 4$

9. Define the sequence a_n by $a_1 = 7$, $a_2 = 3$, and (for $n \ge 3$) $a_n = a_{n-1} - a_{n-2}$. Then $a_{2001} =$

   A. -3 B. -4 C. 3 D. 4 E. 7

10. The area bounded by the quadrilateral whose vertices are the x- and y-intercepts of the lines with equations $4x - 3y = 18$ and $3y - 4x = 12$ is

    A. 37.5 B. 40 C. 42.5 D. 45 E. 48

11. The perpendicular distance between the two parallel lines whose equations are given in problem 10 is

    A. 3 B. 4 C. 5 D. 6 E. 7

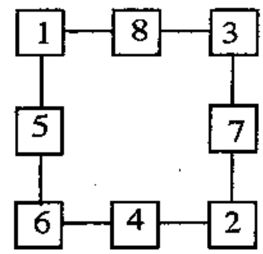
12. A set of 2001 different positive integers are arranged so that $a_1 < a_2 < a_3 < \ldots < a_{2001}$. If the values with odd subscripts are all increased by 1, and those with even subscripts are all decreased by 1, which of the following must be true about the mean μ and median m of the set?

    A. μ decreases by less than 1 and m decreases by 1
    B. μ increases by less than 1 and m increases by 1
    C. μ decreases by less than 1 and m is either unchanged or decreases by 1
    D. μ increases by less than 1 and m is either unchanged or increases by 1
    E. μ and m are both unchanged

13. Right ΔABC (right angle at B) has $m\angle ACB = \alpha$ and $BC = 1$. Right ΔACD (right angle at C) has $m\angle ADC = \alpha$. Then CD =

A. $\cos\alpha$ B. $\sin\alpha$ C. $\tan\alpha$ D. $\cot\alpha$ E. $\csc\alpha$

14. The numbers 1, 2, 3, 4, 5, 6, 7, and 8 are each placed at a different point on a square, either a vertex or a midpoint of a side, in such a way that the sums of the three numbers on each side are all equal (an example with a common sum of 12 is shown at the right). What is the largest possible common sum?



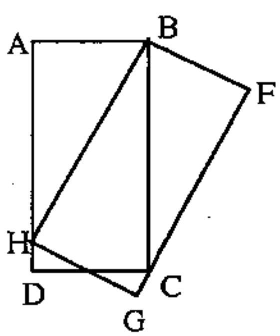
A. 13 B. 14 C. 15 D. 16 E. 17

15. The function f is one-to-one, and the sum of all of the intercepts of the graph of $y = f(x)$ is 5. The sum of all of the intercepts of the graph of $y = f^{-1}(x)$ is

A. 5 B. $\frac{1}{5}$ C. -5 D. $\frac{5}{2}$ E. $\frac{2}{5}$

16. An infinite geometric series s_n $(n \ge 1)$ has sum $\frac{8}{3}$. If the sum of the series t_n for which $t_n = s_n^2$ is twice s_1, the common ratio of the series s_n is

A. $\frac{1}{9}$ B. $\frac{2}{9}$ C. $\frac{1}{3}$ D. $\frac{4}{9}$ E. $\frac{5}{9}$



17. Rectangle ABCD, with AB = 5 in and BC = 13 in, and rectangle BFGH, with BF = 5 in and BH = BC, are placed side by side so that $\overline{BC}$ and $\overline{BH}$ coincide. If rectangle BFGH is rotated around point B until H lies on $\overline{AD}$ and C lies on $\overline{FG}$, the area of the overlap in square inches is closest to

A. 31.2 B. 32.5 C. 33.0 D. 33.8 E. 40.0

18. Define a function f(n) with domain the positive integers as follows: $f(n) = \begin{cases} n + 15 & \text{for n odd} \\ \frac{n}{2} & \text{for n even} \end{cases}$. Let g(n) be the smallest number in the set $\{f(n), f(f(n)), f(f(f(n))), \ldots\}$. How many numbers are in the range of g?

A. 2 B. 3 C. 4 D. 5 E. an infinite number

19. The last two digits of the product of the least common multiple and greatest common factor of $2001! - 1$ and $2^{2001} + 1$ are

A. 01 B. 47 C. 51 D. 53 E. 99

20. The remainder when the polynomial $P(x) = 3x^{500} - 2x^{400} - x^{301} + 2x^{203} - 5x^{101}$ is divided by $x^2 - 1$ is

A. -3 B. $2x - 1$ C. $-2x - 1$ D. $4x + 1$ E. $-4x + 1$

C 1. By rotating the line counterclockwise, it is clear that the maximum slope occurs when the line passes through the origin, producing a slope of $(-2 - 0)/(-3 - 0) = 2/3$.

B 2. $4(1 + i)^{-1} = 4(1 - i)/[(1 + i)(1 - i)] = 4(1 - i)/(1 - i^2) = 2(1 - i)$

B 3. The quadrilateral with the three vertices plus (9,-2) is a right triangle with area 14 plus a trapezoid with area 22. The original triangle's area is this sum minus the area of a right triangle with area 16.

D 4. Since $400 \div 13$ (12 diners plus 1 server) = 30 10/13, we can have 30 full groups of 12 diners. There is room for 10 more people, which makes 9 diners and a server, for a total of $(30)(12) + 9 = 369$ diners.

E 5. C = 1, T = 2, A = Y = 3, M = 4. The A's can't be adjacent, so the only possible order is C,T,A,Y,A, M.

D 6. The equation is equivalent to $2x - 1 = \pm 7$, which has the solutions 4 and -3.

B 7. The runners first meet when the faster one has run 360 m. Each time they meet, they have run one lap between them, so they meet for the 8th time when the faster one has run 2880 m, 120 m short of the start.

E 8. $R = kL/D^2$, so $R_{new} = k(1.5)L/(2.5D)^2 = (6/25)(kL/D^2) = (6/25)(600) = 144$ ohms.

D 9. $\cot C = 2$ means $\cos C = 2 \sin C$, so $\sin 2C = 2 \sin C \cos C = 4 \sin^2 C$. But $\cot^2 C + 1 = 5 = \csc^2 C = 1/\sin^2 C$, so $4 \sin^2 C = 4/5$.

E 10. There are 3 one-scoop cones, 3 two-scoop cones, and 1 three-scoop cone. Of these, four contain chocolate (chocolate, choc-vanilla, choc-strawberry, choc-van-straw), so the answer is 4/7.

A 11. Place the triangles at A(8,0), B(0,0), C(0,6), D(0,5), E(12,0). The hypotenuses have the equations $6x + 8y = 48$ and $5x + 12y = 60$ and intersect at F(3,15/4). DF is $\sqrt{3^2 + (-5/4)^2} = 13/4$.

A 12. Since $(x - a)(x - b)(x - c) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc$, the sum of all of the solutions is -3. The graph indicates one real solution of about 1.1, so the nonreal solutions must sum to about -4.1.

D 13. The equation is equivalent to $(\cos^2 t - \sin^2 t) - (\cos t + \sin t) = 0$ or $(\cos t + \sin t)(\cos t - \sin t - 1) = 0$. The factor $\cos t + \sin t$ is 0 at $3\pi/4$, $7\pi/4$, $-\pi/4$, and $-5\pi/4$, and $\cos t - \sin t = 1$ when $\cos t = 1$ and $\sin t = 0$ $(-2\pi, 0, 2\pi)$ or $\cos t = 0$ and $\sin t = -1$ $(3\pi/2, -\pi/2)$, for a total of nine solutions.

D 14. Since $\log r + \log s = \log (rs)$, $rs = r + s$ with $r, s > 0$ (because of the domain of the log). Solving for s gives $s = r/(r - 1)$. For $r > 1$, this yields $s > 0$, while $r \leq 1$ yields either $s < 0$ or an undefined value.

C 15. The area inside R_1 is 80. The region within one unit of R_2 is bounded by four segments each parallel to a side of R_2 outside the rectangle connected by four quarter circles at the corners, and an inner 4 by 2 rectangle. The area is $20 + \pi + [(6)(4) - (4)(2)] = 36 + \pi$, and the probability is $(36 + \pi)/80$.

E 16. Let a_n = the number of ways of reaching n. You can reach n either by moving one unit from n - 1 or two units from n - 2, so $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 1$ and $a_2 = 2$, the formula yields $a_{10} = 89$.

B 17. Linear programming theory shows that the minimum of a linear function on a polygonal region occurs at a vertex. The vertices of the given region are (6,10), (4,6), (12,6), and (6,3), and the function $f = 2x + y$ assumes its minimum value of 14 at (4,6).

X 18. The first car's distance at time t is 30t, the second's is 40t. Their distances are two sides of a triangle with included angle 135°. By the Law of Cosines, the distance between them is $t\sqrt{2500 + 1200\sqrt{2}}$, and the speed at which they are moving apart is about $\sqrt{4196.8}$ or 65 mph. Due to a misprint, this was not one of the choices, so this question was not graded.

E 19. Since $5^{2000} - 1 = (5^{125} - 1)(5^{125} + 1)(5^{250} + 1)(5^{500} + 1)(5^{1000} + 1)$, a factor of $5^{125} + 1$ or $5^{125} - 1$ is a factor of $5^{2000} - 1$. Since $x^n \pm 1$ is divisible by $x \pm 1$, 6 and 4 (and thus 3 and 2) are factors.

C or D 20. If $\mu = 1$, then since each $x \geq 1$, all values must be 1, so $\mu = 1$ implies $M = m = 1$ as well. This eliminates A, B, and E. For C, a collection with 999 1's, 5 2's, 995 3's, and one 6 satisfies the requirements, and for D, a collection with 1200 1's and 800 6's satisfies them (other choices are possible).

Test #2 **Student Mathematics League Short Solutions** **February 2001**

B 1. The line M has slope 3/2 and x-intercept 3, so the line L has equation $y = (-2/3)x + 2$.

D 2. The squares of the distances are 233, 234, 233, 232, and 234 respectively.

A 3. Replacing x with 2 yields $6 - b = 3b - 6$, so b must be 3.

E 4. The given values are 7, 14, -1, 3, and 4 respectively.

D 5. By the Difference Formula, $\cos(3\pi/2 - t) = \cos(3\pi/2)(\cos t) + \sin(3\pi/2)(\sin t) = 0 + (-1)(\sin t)$.

D 6. Since $4^{3/2} = 8$ and $4^{5/2} = 32$, the required sum is 4.

E 7. Neither of the fractions in the complex fraction can have 0 denominators, eliminating 2 and 1, but $x = -1$ would make the denominator of the complex fraction 0 as well, so it is also eliminated.

B 8. The area of a region made up of two overlapping regions is the sum of the regions less the overlap (which was counted twice), so the answer is $(10)(6) + (4)(15) - (4)(6) = 96$.

C 9. Subtracting the third equation from the first, and k times the third from the second, yields $(k - 1)y = 2$ and $(2 - k)y = 2 - k$. If $k = 2$, the last equation becomes $0 = 0$ and the first equation gives $y = 2$ (so $x = -1$), while if $k \neq 2$, the last equation yields $y = 1$ (and $x = 0$), making $k = 3$.

D 10. The months from January to December have 31, 14, 10, 7, 6, 5, 4, 3, 3, 3, 2, and 2 lucky dates, respectively, for a total of 90 (leap years don't add lucky dates, since 2/29 isn't lucky). The answer is 90/365 or about 0.25. The answer does not change if we count leap year (90/366), four years with one leap year (360/1461), or a full 400-year cycle of the Gregorian calendar (36000/146097).

C 11. The numerator $= x^4 - 2x^3 + x^2 + x^2 - 2x + 1 = (x^2 + 1)(x - 1)^2 = (x + i)(x - i)(x - 1)^2$. Thus the numerator is zero for $x = \pm i$ or 1, but 1 is not a solution, since it is not in the domain of the fraction.

B 12. The condition produces the equation $\sqrt{(x - 3)^2 + (y + 1)^2} = 2\sqrt{(x - 4)^2 + (y - 2)^2}$, which simplifies to $3x^2 + 3y^2 - 26x - 18y + 70 = 0$, the equation of a circle.

E 13. A knight could never say, "I am an alternator" (it would be a lie), so Al is a knave or an alternator. If Al is a knave, "Bo is not an alternator" is false, so Bo is an alternator. If Al is an alternator, "I am an alternator" is true, and his next statement must be false, again making Bo an alternator. Since Bo must be an alternator, his statement "Al is a knave" could be true or false, so E is the only certain choice.

C 14. T must be 1 less than A and C must be 2 less than A, so $T - C = (A - 1) - (A - 2) = 1$.

B 15. If the terms were all different, there would be 8! or 40,320 arrangements. But the two A's can be rearranged in $2! = 2$ ways, and the b's and c's can each be rearranged in $3! = 6$ ways, all without affecting the actual expression, so the total number of arrangements is $8!/(2!3!3!) = 40320/72 = 560$ ways.

A 16. By the Law of Cosines, $8^2 = 11^2 + 9^2 - 2(99)\cos A$, and $CM^2 = 11^2 + 4.5^2 + 99 \cos A$. Subtracting twice the second equation from the first yields $64 - 2CM^2 = -80.5$, so $CM^2 = 289/4$ and $CM = 8.5$.

B 17. $a_1 = 1$, $a_3 = 3$, $a_5 = 7$, $a_7 = 15$, and it is easy to show that $a_{2n-1} = 2^n - 1$, so $a^{2001} = 2^{1001} - 1$. Then $\sqrt[2001]{a_{2001}} = (2^{1001} - 1)^{1/2001}$ or approximately $2^{1001/2001}$, which is closest to $\sqrt{2}$.

B 18. The original collection has the form _ _ _ 3 _ _ _, with either two or three 4's above the 3. If there were only two 4's, 4 couldn't be the unique mode, since the three blanks before the 3 could only be filled with one 1 and two 2's, or two 1's and one 2. Thus the collection must be _ _ _ 3 4 4 4, with the blanks filled with 1 2 2 or 2 2 3 (1 1 1, 2 3 3, 1 3 3, or 2 2 2 means 4 is not the unique mode, and 1 2 3 means 2 isn't the new mode). But 2 2 3 means 3 is the median of the new collection, so the original must have been 1 2 2 3 4 4 4, making the mean of the new collection (1+2+2+2+2+3+4+4+4)/9 = 8/3.

E 19. Since 2001 = 3(23)(29) and $r^2 - s^2 = (r - s)(r + s)$, r - s must be 1, 3, 23, or 29, making r + s 2001, 667, 87, or 69, giving four solutions: (1001,1000), (335,332), (55,32), and (49,20).

D 20. The number 8 cannot be a sum, since it can only be written as 1 + 2 + 5 or 1 + 3 + 4, so there is no third side for the triangle; 13 cannot be a sum, since it is only 2 + 5 + 6 or 3 + 4 + 6. For the numbers between 8 and 13, the following solutions exist (reading clockwise form the top of the triangle): 9: 1,5,3,4,2,6; 10: the given example; 11: 2,3,6,1,4,5; 12: 5,1,6,2,4,3.

Test #3 **Student Mathematics League Short Solutions** **March-April 2001**

B 1. r = -1, s = -7, and t = 44, so the desired order is s < r < t.

B 2. For $1 \le x \le 2$, $|x - 3| = 3 - x$, $|1 - x| = x - 1$, and $|4 - 2x| = 4 - 2x$.

A 3. The sum of the intercepts of y = 2x + b is b/2. The sum of the intercepts of y = 2x + b + k is (b + 4) + (-b/2 - 2) = b/2 + 2 = k + 2.

D 4. Carlos can make 4(5) meat, 4(3) cheese, and 4(5)(3) meat and cheese sandwiches. The total = 92.

C 5. Let a = George's investment. Then 0.9(1.1a - 150) + 150 = 1.04a, and a = 300, so his gain is $12.

C 6. The months from January to December have 31, 14, 10, 7, 6, 5, 4, 3, 3, 3, 2, and 2 lucky dates, respectively, so the answer is 3 months (September, October, and November).

C 7. A < 5, so we have 4M + 4T = YC. Since we want Y = 9, M and T must be 8 and 7, and YC = 95.

E 8. $4 - x \ge 0$ implies $x \le 4$, and $4 - \sqrt{4 - x} \ge 0$ implies $\sqrt{4 - x} \le 4$, $4 - x \le 16$, and $x \ge -12$.

B 9. The sequence has $a_{n+6} = a_n$, so $a_{2001} = a_{3+6(333)} = a_3 = -4$.

A 10. The region consists of 4 right triangular regions with right angles at (0,0) and areas 6, 9, 9, and 13.5.

D 11. The region is a trapezoid, and the distance is its altitude. Since the two bases of the trapzoid have lengths 5 and 7.5, the altitude = 2(37.5)/(5 + 7.5) = 6.

D 12. Of the a_i's, 1001 increase by 1 and 1000 decrease by 1, so μ increases by 1/2001, eliminating A, C, and E. If a_{1001} (the median) = $a_{1002} - 1$, then these two elements switch values, so the median is unchanged. If $a_{1001} < a_{1002} - 1$, then the median increases by 1.

E 13. In ΔABC, $\sec \alpha = AC/1 = AC$. In ΔACD, $\cot \alpha = CD/\sec \alpha$, so $CD = (\cot \alpha)(\sec \alpha) = \csc \alpha$.

C 14. Since 17 isn't the sum of three different numbers ≤ 8 including a 1, 17 is impossible. The only such sum adding to 16 is 1 + 7 + 8, so if such a square exists, one side is 7, 1, 8. The only sum adding to 16 including a 2 is 2 + 6 + 8, so an adjacent side must be 8, 2, 6. But the only remaining sum including a 6 is 3 + 6 + 7, and 6 and 7 are diagonally opposite. The square with values reading clockwise from a corner of 3, 4, 8, 1, 6, 2, 7, 5 has common sum 15.

A 15. A function and its inverse are symmetric across y = x, their x- and y-intercepts are interchanged, and the sum is the same.

C 16. The first series' sum = $s_1/(1 - r) = 8/3$, and the second series' sum = $s_1^2/(1 - r^2) = 2s_1$, so $s_1/[(1 - r)(1 + r)] = 2$. Thus 8/3 = 2(1 + r), and r = 1/3.

D 17. ΔBCF is a right triangle with BC = 13 and BF = 5, so CF = 12. Then CG = 1, and by similar triangles, CM (where M is the intersection of $\overline{CD}$ and $\overline{GH}$) = 13/5. By the same argument, HM = 13/5, and the overlap has area 2(1/2)(13)(13/5) = 33.8 (two right trangles with legs 13 and 13/5).

D 18. Every power of 2 reduces to 1; 3 cycles back to 3 (but no smaller); 5 cycles back to 5 (but no smaller); 6, 9, and 12 all cycle to 3; 7 cycles to 7 (but no smaller); 10 cycles to 5; 11, 13, and 14 all cycle to 7; 15 cycles to 15 (but no smaller); all higher numbers cycle to one of these for a total of 5: {1, 3, 5, 7, 15}.

B 19. The product of the LCM and GCF of x and y is xy, so the last two digits of $(2001! - 1)(2^{2001} + 1)$ are what we seek. 2001! ends in many zeros, so 2001! - 1 ends in 99. Looking at the last two digits of powers of 2 shows that 2^{21}, 2^{41}, 2^{61}, …, 2^{2001} all end in 52, so $2^{2001} + 1$ ends in 53, and the required product has the same last two digits as (99)(53) = (100 - 1)(53) = 5247, so the answer is 47.

E 20. By the division algorithm, $P(x) = Q(x)(x^2 - 1) + R(x)$, where the degree of R(x) must be ≤ 1, so $R(x) = mx + b$. But $P(1) = -3 = Q(1)(0) + R(1)$, and $P(-1) = 5 = Q(-1)(0) + R(-1)$, so $-3 = m + b$ and $5 = -m + b$, which means $m = -4$ and $b = 1$.