SMLC w/SOL'N ~~2003~~-2004

Test #2 **AMATYC Student Mathematics League** **February/March 2004**

1. A stock loses 60% of its value. What must the percent of increase be to recover all of its lost value?

A. 60% B. 120% C. 150% D. 200% E. 300%

2. Which of the following is NOT a factor of $x^4 - 4x^3 - x^2 + 16x - 12$?

A. $x - 2$ B. $x + 2$ C. $x - 1$ D. $x + 1$ E. $x - 3$

3. The library in Johnson City has between 1000 and 2000 books. Of these, 25% are fiction, 1/13 are biographies, and 1/17 are atlases. How many books are either biographies or atlases?

A. 240 B. 250 C. 270 D. 280 E. 300

4. A tricimal is like a decimal, except the digits represent fractions with powers of 3 instead of 10. For example, $16/27 = 1/3 + 2/9 + 1/27 = 0.121$ as a tricimal. How is $77/81$ expressed as a tricimal?

A. 0.950617 B. 0.2012 C. 0.1211 D. 0.1111 E. 0.2212

5. The function $P(t) = \cos 8t$ can be written as sums and differences of powers of cos t. When P(t) is written that way, what is the coefficient of $(\cos t)^3$?

A. 0 B. 1 C. -1 D. 2 E. -2

6. If $\log_a b = 64$, find $\log_{a^2} b^3$.

A. 16 B. 48 C. 128/3 D. 96 E. 512

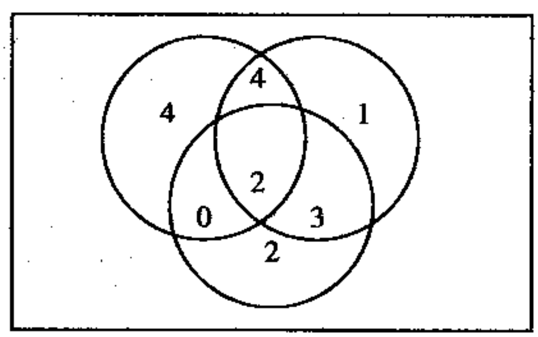
7. The number 877530p765q6 is divisible by both 8 and 11, with p and q both digits from 0 to 9. The number is also divisible by

A. 7 B. 12 C. 16 D. 18 E. not enough information to know

8. Teams A and B play a series of games; whoever wins two games first wins the series. If Team A has a 70% chance of winning any single game, what is the probability that Team A wins the series?

A. 0.616 B. 0.637 C. 0.657 D. 0.700 E. 0.784

9. The Venn diagram at the right represents sets A, B and C (not necessarily in that order). Depending on how the diagram is labeled, how many different answers are possible for the number of elements in the set A - B ? (Note: A - B is all elements which are in A but not in B)



A. 2 B. 3 C. 4 D. 5 E. 6

10. A fixed point for a function $y = f(x)$ is a real number r such that $f(r) = r$. How many of the following functions must have a fixed point?

polynomial function of odd degree > 1
polynomial function of even degree > 0
trigonometric function $y = A \sin Bx + D$
rational function $y = (x - a)/(x - b)$

A. 0 B. 1 C. 2 D. 3 E. 4

11. Which of the following is the identity function $f(x) = x$ for all real numbers?

A. $e^{\ln x}$ B. $\ln e^x$ C. $\sin(\arcsin x)$ D. $\arctan(\tan x)$ E. $\sqrt{x^2}$

12. A circular table is pushed into a corner of a rectangular room so that it touches both walls. A point on the edge of the table between the two points of contact is 2 inches from one wall and 9 inches from the other wall. What is the radius of the table?

A. 5 inches B. 12 inches C. 15 inches D. 17 inches E. 20 inches

13. In ΔABC, $\angle C = 90°$ and $\cos \angle A = 4/5$. If D is the midpoint of side AC, find $\cos \angle CDB$.

A. $\frac{2\sqrt{13}}{13}$ B. $\frac{5}{9}$ C. $\frac{\sqrt{5}}{4}$ D. $\frac{2}{5}$ E. $\frac{3}{5}$

14. Enrique walks along a level road and then up a hill. At the top he immediately turns and walks back to his starting point. He walks 4 mph on level ground, 3 mph uphill, and 6 mph downhill. If the entire walk takes 6 hours, how far does he walk?

A. 16 mi B. 20 mi C. 24 mi D. 28 mi E. 32 mi

15. If $x^2 = x + 3$, then $x^3 =$

A. $x + 6$ B. $4x + 3$ C. $4x^2 + 3$ D. $x^2 + 3x + 3$ E. $x^2 + 27$

16. A bag holds 5 cards identical except for color. Two are red on both sides, two are black on both sides, and one is red on one side and black on the other. If you pick a card at random and see that the only side you can see is red, what is the probability that the other side is also red?

A. 1/2 B. 2/3 C. 3/4 D. 4/5 E. 5/6

17. The set S contains the number 2, and if it contains the number n, it also contains 3n and n + 5 (assume S contains only numbers produced by these rules). Which of the following is NOT in S?

A. 2000 B. 2001 C. 2002 D. 2003 E. 2004

18. Let $f(x) = ax + b$, with $b < a$ both positive integers. If for positive integers p and q, $f(p) = 18$ and $f(q) = 39$, what is the value of b?

A. 1 B. 3 C. 4 D. 7 E. 8

19. In ΔSBC, SB = 12, BC = 15, and SC = 18. Let O be the point for which BO bisects angle SBC and CO bisects angle SCB. If M and L are on sides SB and SC respectively so that ML is parallel to side BC and contains point O, what is the perimeter of ΔSML?

A. 24 B. 27 C. 30 D. 32 E. 36

20. Ed has four children, Al, Bo, Cy, and Di (in order oldest to youngest). Bo is 4 years older than Cy and 12 years older than Di. This year Ed notices that he is twice as old as Bo, and the sum of the squares of the children's ages equals the square of Ed's age. If Di just became a math teacher, what is the sum of the children's ages?

A. 48 B. 76 C. 100 D. 128 E. 148

C 1. Suppose the original value was \$100. A 60% loss reduces it to \$40. An increase of \$60 is 150%.

D 2. The expression $x^4 - 4x^3 - x^2 + 16x - 12 = (x^4 - 4x^3 + 4x^2) - (5x^2 - 16x + 12) = x^2(x - 2)^2 - (5x - 6)(x - 2)$ $= (x - 2)(x^2(x - 2) - 5x + 6) = (x - 2)(x(x - 1)^2 - 6(x - 1)) = (x - 2)(x - 1)(x - 3)(x + 2)$.

A 3. The number of books in the library must be divisible by 4, 13, and 17. Thus the number of books is a multiple of $(4)(13)(17) = 884$. The only multiple of 884 between 1000 and 2000 is 1768, so the number of biographies or atlases must be $1768/13 + 1768/17 = 136 + 104 = 240$.

E 4. The fraction $77/81 = 2/3 + 23/81 = 2/3 + 2/9 + 5/81 = 2/3 + 2/9 + 1/27 + 2/81 = 0.2212$.

A 5. $\text{Cos } 8t = 2\cos^2 4t - 1 = 2(2\cos^2 2t - 1)^2 - 1 = 2(2(2\cos^2 t - 1)^2 - 1)^2 - 1$, which involves only even powers of cos t, so the coefficient of $(\cos t)^3$ is 0.

D 6. If $\log_a b = 64$, then $a^{64} = b$, $(a^2)^{32} = b$, and $(a^2)^{96} = b^3$. Thus $\log_{a^2} b^3 = 96$.

B 7. A number is divisible by 8 only if its last three digits are, so q must be 3 or 7. It is easy to determine that q cannot equal 7, so $p = 3$ and $q = 3$. The resulting number is divisible by 3, but not by 7, 9, or 16.

E 8. Team A wins the series if it wins the first two games or if it wins exactly one of the first two games and then the third game. This probability is $(0.7)(0.7) + (0.7)(0.3)(0.7) + (0.3)(0.7)(0.7) = 0.784$.

C 9. The number of elements in A - B could be the sum of any two numbers inside one circle and outside another. These are $4 + 4$, $4 + 1$, $1 + 3$, $3 + 2$, $2 + 0$, and $0 + 4$, which yield four distinct sums.

C 10. Graphically, a function has a fixed point if it intersects the line $y = x$. Since an odd degree polynomial approaches $+\infty$ in one direction and $-\infty$ in the other, it must cross this line. An even degree polynomial could lie entirely above the line; a sine curve must cross it; a rational function could have the line $y = x$ as an asymptote. Thus the answer is two.

B 11. The function $e^{\ln x}$ is undefined for $x \le 0$; sin(arcsin x) is undefined for $|x| > 1$; arctan(tan x) = x only if x is between $-\pi/2$ and $\pi/2$; $\sqrt{x^2} = |x|$, not x; so the answer must be $\ln e^x$.

D 12. Let r be the radius of the table; then $(r - 2)^2 + (r - 9)^2 = r^2$, since the given point is one vertex of a right triangle. Then $r^2 - 22r + 85 = (r - 5)(r - 17) = 0$, and $r > 9$ implies $r = 17$.

A 13. The given triangle must have $AC = 4x$, $BC = 3x$, and $AB = 5x$. Then $DC = 2x$, $BD = x\sqrt{13}$, and $\cos \angle CDB = 2/\sqrt{13}$.

C 14. Let L = the distance Enrique walks on the level and H = the distance he walks uphill. Then $L/4 + H/3 + H/6 = 6$ hr. But then $L + 2H = 24$, and $L + 2H$ is his total distance.

B 15. If $x^2 = x + 3$, then $x(x^2) = x(x + 3) = x^2 + 3x = (x + 3) + 3x = 4x + 3$.

D 16. There are 5 possible red sides. Of those sides, 4 have a red side on the other side, so the required probability is 4/5.

A 17. From the rules, if n is not a multiple of 5, then neither is 3n or n + 5. Since S starts out containing 2 (not a multiple of 5), S contains no multiples of 5, hence not 2000. For the other numbers, $2001 = 3(2 + 5(133))$, $2002 = 2 + 5(400)$, $2003 = 3(3(2 + 5(44))) + 5$, and $2004 = 3(3(3(2))) + 5(390)$.

C 18. We know $ap + b = 18$ and $aq + b = 39$, so $a(q - p) = 21$. Thus $a = 1, 3, 7$, or 21. But $a \neq 1$ (since then $b = 0$), $a \neq 21$ (since then $b < 0$), and $a \neq 3$ (since then $b = 0$). Thus $a = 7$, $b = 4$, $p = 2$, $q = 5$.

C 19. By definition of bisect, $\angle OBM = \angle OBC$ and $\angle OCL = \angle OCB$. But $\angle OBC = \angle BOM$ and $\angle OCB = \angle COL$ (by alternate interior angles). Thus $\angle OBM = \angle BOM$ and $\angle OCL = \angle COL$, so triangles OBM and OCL are isosceles with MB = MO and LC = LO. The perimeter of $\Delta SML = SB + SC = 30$.

E 20. From the given information, $4B^2 = A^2 + B^2 + (B - 4)^2 + (B - 12)^2$, so $B^2 + 32B - 160 = A^2$. Completing the square yields $(B + 16)^2 - A^2 = 416$. The only positive integers satisfying this equation are A = 103, B = 89 (making Ed 178); A = 50, B = 38; A = 22, B = 14 (making Di 2 yrs old). Thus the only physically possible solution is: Al is 50, Bo is 38, Cy is 34, Di is 26, with total ages 148.

1. If L has equation $ax + by = c$, M is its reflection across the y-axis, and N is its reflection across the x-axis, which of the following must be true about M and N for all nonzero choices of a, b, and c?

A. the x-intercepts are equal B. the y-intercepts are equal C. the slopes are equal
D. the slopes are opposite E. the slopes are reciprocals

2. A collection of coins is made up of an equal number of pennies, nickels, dimes, and quarters. What is the largest possible value of the collection which is less than \$2?

A. \$1.64 B. \$1.78 C. \$1.86 D. \$1.89 E. \$1.99

3. When the polynomial $P(x)$ is divided by $(x-2)^2$, the remainder is $3x - 3$. What is the remainder when $(x-1)P(x)$ is divided by $(x-1)(x-2)^2$?

A. $3x - 3$ B. $3x^2 - 6x + 3$ C. 3 D. $x - 1$ E. $x - 2$

4. If $f(x) = 3x - 2$, find $f(f(f(3)))$. A. 19 B. 55 C. 75 D. 107 E. 163

5. What is the remainder when $x^3 - 2x^2 + 4$ is divided by $x + 2$?

A. -12 B. 0 C. 4 D. 6 E. 12

6. Let p be a prime number and k an integer such that $x^2 + kx + p = 0$ has two positive integer solutions. What is the value of $k + p$?

A. 1 B. -1 C. 0 D. 2 E. -2

7. What is the least number of prime numbers (not necessarily different) that 3185 must be multiplied by so that the product is a perfect cube?

A. 1 B. 2 C. 3 D. 4 E. 5

8. Two adjacent faces of a three-dimensional rectangular box have areas 24 and 36. If the length, width, and height of the box are all integers, how many different volumes are possible for the box?

A. 2 B. 3 C. 4 D. 5 E. 6

9. $(\tan t - \sin t \cos t)/(\tan t) =$

A. $\sin t$ B. $\cos t$ C. $\sin^2 t$ D. $\cos^2 t$ E. 1

10. The counting numbers are written in the pattern at the right. Find the middle number of the 40^{th} row.

			1			
		2	3	4		
	5	6	7	8	9	
10	11	12	13	14	15	16

A. 1561 B. 1641 C. 1559 D. 1639 E. 1483

11. The solution set of $x^2 - 3x - 18 \geq 0$ is a subset of the solution set of which of the following inequalities?

A. $x^2 - x - 20 \geq 0$ B. $(x - 4)/(x + 3) \geq 0$ C. $x^2 - 8x + 14 \geq 0$

D. both B and C E. all of A, B, and C

12. If $2a - 4b = 128b^3 - 16a^3$ and $a \neq 2b$, find $a^2 + 2ab + 4b^2$.

A. -1/8 B. -1/2 C. 1/2 D. 1/8 E. 2

13. Square ABCD is inscribed in circle O (that is, A, B, C, and D all lie on the circle) and its area is a. Square EFGH is inscribed in a semicircle of circle O (that is, E and F lie on a diameter and G and H lie on the circle). What is the area of square EFGH?

A. a/5 B. 2a/5 C. a/3 D. a/2 E. 3a/5

14. Consider all arrangements of the letters AMATYC with either the A's together or the A's on the ends. What fraction of all possible such arrangements satisfies these conditions?

A. 1/5 B. 2/15 C. 1/3 D. 2/5 E. 3/5

15. The year 2003 is prime, but its reversal, 3002, is not. In fact, 3002 is the product of exactly three different primes. Let N be the sum of these three primes. How many other positive integers are the products of exactly three different primes with this sum N?

A. 0 B. 1 C. 2 D. 3 E. 4

16. In a group of 30 students, 25 are taking math, 22 English, and 19 history. If the largest and smallest number who could be taking all three courses are M and m respectively, find M + m.

A. 17 B. 19 C. 22 D. 23 E. 25

17. A boat with an ill passenger is 7 1/2 mi north of a straight coastline which runs east and west. A hospital on the coast is 60 miles from the point on shore south of the boat. If the boat starts toward shore at 15 mph at the same time an ambulance leaves the hospital at 60 mph and meets the ambulance, what is the total distance (to the nearest 0.5 mile) traveled by the boat and the ambulance?

A. 60.5 B. 61 C. 61.5 D. 62 E. 62.5

18. If each letter in the equation $\sqrt{AMATYC} = MYM$ represents a different decimal digit, find T's value.

A. 3 B. 4 C. 5 D. 6 E. 7

19. If a, b, c, and d are nonzero numbers such that c and d are solutions of $x^2 + ax + b = 0$ and a and b are solutions of $x^2 + cx + d = 0$, find $a + b + c + d$.

A. -2 B. -1 C. 0 D. 1 E. 2

20. Al and Bob are at opposite ends of a diameter of a silo in the shape of a tall right circular cylinder with radius 150 ft. Al is due west of Bob. Al begins walking along the edge of the silo at 6 ft per second at the same moment that Bob begins to walk due east at the same speed. The value closest to the time in seconds when Al first can see Bob is

A. 46 B. 47 C. 48 D. 49 E. 50

C 1. M has equation $-ax + by = c$, and N has equation $ax - by = c$. The x-intercepts are $-c/a$ and c/a, the y-intercepts are c/b and $-c/b$, and the slopes are both a/b, so only C is true.

A 2. One penny, nickel, dime, and quarter are worth 41¢. The largest multiple of 41 less than 200 is 164.

B 3. By the Division Algorithm, $P(x) = Q(x)(x-2)^2 + (3x-3)$, where $Q(x)$ is the quotient. Then $(x-1)P(x) = Q(x)(x-1)(x-2)^2 + (x-1)(3x-3)$. Thus the new remainder is $3x^2 - 6x + 3$.

B 4. Since $f(3) = 7$, $f(7) = 19$, and $f(19) = 55$, the answer is B.

A 5. By the Remainder Theorem, the remainder is $(-2)^3 - 2(-2)^2 + 4$, or -12.

B 6. For p prime and positive solutions, the quadratic must factor into $(x-1)(x-p)$, so that $k = -(p+1)$. Thus the sum of k and p is -1.

E 7. The number 3185 factors into (5)(7)(7)(13). To make it a perfect cube, multiply by 5, 5, 7, 13, and 13.

E 8. The common edge must be a factor of both 24 and 36. Thus the possible dimensions of the box are 1x24x36, 2x12x18, 3x8x12, 4x6x9, 4x6x6, and 2x3x12. All six volumes are different.

C 9. The given fraction equals $1 - \sin t \cos t \cot t = 1 - \cos^2 t = \sin^2 t$.

A 10. Notice that n^2 is at the end of row n, so the middle term of row 40 must be $((39^2 + 1) + 40^2)/2 = 1561$.

C 11. The given inequality has solution $(-\infty,-3] \cup [6,+\infty)$. But -3 does not satisfy either A or B (because of division by zero), so the answer must be C. Note that the solution of C is $(-\infty, 4-\sqrt{2}] \cup [4+\sqrt{2},+\infty)$.

A 12. Since $128b^3 - 16a^3 = 2(4b-2a)(16b^2 + 8ab + 4a^2)$, then $2a - 4b = 2(4b-2a)(16b^2 + 8ab + 4a^2)$, and $-1 = 8(a^2 + 2ab + 4b^2)$, so $a^2 + 2ab + 4b^2 = -1/8$.

B 13. Let s be the side length of square EFGH. Then $s^2 + (s/2)^2 = r^2$, so $r = \dfrac{\sqrt{5}}{2}s$. Then a side of square ABCD $= 2r/\sqrt{2} = \dfrac{\sqrt{10}}{2}s$, and the area of ABCD $= a = (5/2)s^2$, so the area of EFGH $= s^2 = (2/5)a$.

D 14. The number of arrangements of AMATYC is $6!/2 = 360$ (division by 2 accounts for the duplicate A's). If the A's are on the ends, there are $4! = 24$ ways to place the remaining 4 letters, and if the A's are together, there are 5 ways to place the A's (consider them as a single object) and 4! ways to place the remaining letters for a total of 120 arrangements. The required probability is $(24 + 120)/360 = 2/5$.

C 15. Factoring 3002 yields 2(19)(79), so the prime factors add to 100. For three primes to add to 100, one must be a 2, and the only other possibilities for 98 are 31 + 67 and 37 + 61.

E 16. Clearly $M = 19$. To find m, assume the math and English students overlap as little as possible. Since there are 5 students not taking math, the smallest overlap is 17. Then if the history students overlap the 8 math students not taking English and the 5 English students not taking math, there must be $19 - 5 - 8 = 6$ history students also taking both math and English. Thus $m = 6$ and $M + m = 25$.

E 17. Let x be the distance from the hospital to the meeting point. Then $x/60 = \dfrac{\sqrt{7.5^2 + (60-x)^2}}{15}$ (using time is distance over speed). This becomes $(x-50)(x-78) = 0$, whose only valid solution is 50. The boat then travels 12.5 miles for a total for the two vehicles of 62.5 miles.

E 18. Examining the equation $AMATYC = MYM^2$ shows $316 < \sqrt{100000} \le MYM \le \sqrt{999999} < 999$. A quick check of the numbers 323, 343, 353, etc. shows MYM = 363 and AMATYC = 131769, so T = 7.

A 19. The given conditions mean that $x^2 + ax + b = x^2 - (c+d)x + cd$, and $x^2 + cx + d = x^2 - (a+b)x + ab$. Thus $-a = c + d$, $b = cd$, $-c = a + b$, and $d = ab$. Hence $b = abc$ and $ac = 1$. Also, $a + c + d = 0 = a + b + c$, so $b = d$, which means $c = a = 1$. This means that $b = d = -2$, so $a + b + c + d = -2$.

C 20. After t sec, Al's central angle is $6t/150$ rad, while Bob's distance from the silo is 6t. Al first sees Bob when his location is the point of tangency of the line to Bob's position. This forms a right triangle, so that $\cos(\pi - 6t/150) = 150/(150 + 6t)$. This equation is satisfied by $t = 48.0075$.